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The Uniqueness of the Inverse Obstacle Scattering Problem with Transmission Boundary Conditions

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Abstract—In this paper, we consider the uniqueness of the interior wavenumber for the inverse obstacle scattering problem with transmission boundary conditions. We show that the interior wavenumber is uniquely determined by the far field pattern. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Inverse obstacle scattering, Transmission boundary value problem.

1. INTRODUCTION

The problem of uniqueness in inverse obstacle scattering theory is of central importance both for the theoretical study and the implementation of numerical algorithms. There is a large variety of possible inverse problems arising from different given data and boundary conditions (see [1,2]), and many results have been obtained (see [1,3–10]). In this paper, we use an idea of Hettlich's [6] to show the uniqueness of the interior wavenumber for the inverse transmission problem. In Section 2, we derive a relationship between solutions of the transmission problem for which the far field patterns coincide. Then, by a proof by contradiction, we establish uniqueness by constructing special singular solutions in Section 3.

2. THE INVERSE PROBLEM

The scattering of plane acoustic waves by a penetrable obstacle D with density ρ_D and speed of sound c_D differing from the density ρ and the speed of sound c in the surrounding medium $\mathbb{R}^3 \setminus D$ leads to a transmission problem for the Helmholtz equations [10] (see also [5]):

$$\begin{aligned} \Delta u + k^2 u = 0, &\qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \Delta v + k_D^2 v = 0, &\qquad \text{in } D, \end{aligned} \tag{2.1}$$

with the transmission boundary conditions

$$\begin{aligned} u &= v, \\ \frac{\partial u}{\partial \nu} &= \lambda \frac{\partial v}{\partial \nu}, \end{aligned} \qquad \text{on } \partial D, \tag{2.2}$$

where ν denotes the outward normal to ∂D, $k_D^2 = k^2 c^2 / c_D^2 \neq k^2$ and $\lambda = \rho/\rho_D$. Here, u is the superposition of the scattered wave u^s and the given incident plane wave $u^i = e^{ikx\cdot d}$, i.e.,

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$u = u^s + u^i$, and the scattered wave u^s is required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial \nu} - iku^s\right) = 0, \qquad r = |x| \tag{2.3}$$

uniformly in all directions $\hat{x} = x/|x|$. This condition ensures the uniqueness of solution for the exterior boundary value problem and leads to an asymptotic expansion of the form

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \qquad |x| \to \infty. \tag{2.4}$$

The function $u_\infty(\hat{x})$ is the far field pattern (also called scattering amplitude) of the scattered wave.

From Green's representation theorem [1, Theorem 2.5] and the asymptotic behaviour of the fundamental solution

$$\Phi(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \qquad x \neq y, \tag{2.5}$$

we have a representation of the far field pattern in the form

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\left\{u^s(y)\frac{\partial}{\partial \nu_y}e^{-ik\hat{x}\cdot y} - \frac{\partial u^s}{\partial \nu_y}(y)e^{-ik\hat{x}\cdot y}\right\} ds_y.$$

The existence of solutions and well-posedness for the transmission problem can be established by boundary integral equation methods [1].

Throughout the current paper, we assume the boundary ∂D of the scatterer to be of class C^2. In the sequel, in order to indicate the dependence on the incident direction d, we will write $u = u(\cdot; d)$, $u^i = u^i(\cdot; d)$, $u^s = u^s(\cdot; d)$, and $u_\infty = u_\infty(\cdot; d)$, respectively.

The inverse problem in which we are interested is: given the far field pattern $u_\infty(\cdot; d)$ for all incident directions d, the shape ∂D of the scatterer, and the exterior wavenumber k in $\mathbb{R}^3 \setminus D$, determine the interior wavenumber k_D.

An important preliminary result is the one-to-one correspondence between radiating waves and their far field patterns. By Rellich's lemma [1], we have the following lemma.

LEMMA 2.1. *Assume the bounded set D is the open complement of an unbounded domain and let $u \in C^2(\mathbb{R}^3 \setminus \overline{D})$ be a radiating solution to the Helmholtz equation for which the far field pattern vanishes identically. Then, $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.*

Next, we define weak solutions of the transmission problem which will be used in our approach. By Green's theorem in the ball $\Omega_R = \{x \in \mathbb{R}^3; |x| < R\}$ with $\overline{D} \subseteq \Omega_R$, the solution of problem (2.1),(2.2) satisfies the variational equation

$$\int_{\Omega_R}\left(\mu\nabla u\nabla\varphi - K^2u\varphi\right)dx = \int_{\partial\Omega_R}\frac{\partial u}{\partial \nu}\varphi\, ds, \qquad \varphi \in H^1(\Omega_R), \tag{2.6}$$

with

$$\mu(x) = \begin{cases} 1, & x \in \mathbb{R}^3 \setminus D, \\ \lambda, & x \in D, \end{cases} \tag{2.7}$$

and

$$K^2(x) = \begin{cases} k^2, & x \in \mathbb{R}^3 \setminus D, \\ \lambda k_D^2, & x \in D, \end{cases} \tag{2.8}$$

where $H^1(\Omega_R)$ denotes the Sobolev space of order 1, and analogously $H^s(\partial\Omega_R)$ for $s \in \mathbb{R}$.

Let $L : H^{1/2}(\partial\Omega_R) \longrightarrow H^{-1/2}(\partial\Omega_R)$ be the Dirichlet to Neumann map, i.e.,

$$L\psi = \frac{\partial \psi_1}{\partial \nu}, \tag{2.9}$$

where ψ_1 is the uniqueness solution of the exterior Dirichlet problem

$$\begin{aligned} \Delta\psi_1 + k^2\psi_1 &= 0, && \text{in } \mathbb{R}^3 \setminus \overline{\Omega}_R, \\ \psi_1 &= \psi, && \text{on } \partial\Omega_R. \end{aligned} \tag{2.10}$$

By L_0, we denote the corresponding operator with wavenumber $k = 0$. Both operators can be explicitly expressed in terms of spherical harmonics and Bessel functions. We can prove that $-L_0$ is a strictly coercive sesquilinear form, i.e.,

$$-\langle L_0\psi, \psi\rangle \geq c\|\psi\|^2_{H^{1/2}(\partial\Omega_R)}, \qquad \text{for all } \psi \in H^{1/2}(\partial\Omega_R), \tag{2.11}$$

where $\langle , \rangle$ denotes the dual bracket and c is a positive constant. From Theorem 2.1 of [12], $L - L_0$ is a compact operator from $H^{1/2}(\partial\Omega_R)$ to $H^{-1/2}(\partial\Omega_R)$.

DEFINITION. *A function $u \in H^1(\Omega_R)$ is a weak solution of the transmission problem (2.1),(2.2) if u satisfies the following variational equation in Ω_R:*

$$\int_{\Omega_R} (\mu\nabla u\nabla\overline{\varphi})\,dx - K^2u\overline{\varphi} - \langle Lu, \varphi\rangle = \int_{\partial\Omega_R}\left(\frac{\partial u^i}{\partial\nu} - Lu^i\right)\overline{\varphi}\,ds, \tag{2.12}$$

for all $\varphi \in H^1(\Omega_R)$.

LEMMA 2.2. *Let μ, K, k, k_D, λ be defined as in (2.2), (2.7), and (2.8). Then, for any $f \in H^1(\Omega_R)$, there exists a unique solution $u \in H^1(\Omega_R)$ which satisfies*

$$\int_{\Omega_R} \left(\mu\nabla u\nabla\overline{\varphi} - K^2u\overline{\varphi}\right)\,dx - \langle Lu, \varphi\rangle = (f, \varphi)_{H^1(\Omega_R)}, \tag{2.13}$$

for all $\varphi \in H^1(\Omega_R)$.

Lemma 2.2 can be proved by taking $\lambda = 0$ in Theorem 2.1 in [6].

To establish uniqueness of the inverse problem from the knowledge of the far field patterns for all incident directions d, we need to establish the relationship between two solutions of the transmission problem for which the far field patterns coincide.

First, we assume D_1 and D_2 are two obstacles, k is the exterior wavenumber, k_j are the interior wavenumbers, and λ_j are the jump parameters ($j = 1, 2$). By

$$\lambda_j = \frac{\rho}{\rho_j}, \qquad k_j^2 = \frac{k^2c^2}{c_j^2},$$

observe that if $\lambda_1 = \lambda_2$, and hence, $\rho_1 = \rho_2$, then $c_1 = c_2$, and hence, $k_1 = k_2$.

Now assume the conditions of Lemma 2.1. Denote by $\Omega_c \subseteq \mathbb{R}^3$ the unbounded connected part of $\mathbb{R}^3 \setminus (\overline{D_1 \cup D_2})$, and by $\Omega_0 = \mathbb{R}^3 \setminus \overline{\Omega}_c$ its complement, then we have the following two lemmas.

LEMMA 2.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\overline{\Omega}_0 \subseteq \Omega$. For a fixed incident direction d, we assume that $u_{\infty,1}(\cdot; d) = u_{\infty,2}(\cdot; d)$. Then,*

$$\int_{D_1} \left[(1-\lambda_1)\nabla u_1\nabla\tilde{v}_2 + \left(k_1^2 - k^2\right)u_1\tilde{v}_2\right]dx = \int_{D_2}\left[(1-\lambda_2)\nabla u_1\nabla\tilde{v}_2 + \left(k_2^2 - k^2\right)u_1\tilde{v}_2\right]dx, \tag{2.14}$$

for every solution $\tilde{v}_2 \in H^1(\Omega)$ of the variational equation

$$\int_\Omega \left[-\tilde{\mu}_2\nabla\tilde{v}_2\nabla\varphi + \tilde{K}_2^2\tilde{v}_2\varphi\right]dx = 0, \qquad \varphi \in H_0^1(\Omega), \tag{2.15}$$

where the coefficients $\tilde{\mu}_2$, $\tilde{K}_2 \in L^\infty(\Omega)$ satisfy $\tilde{\mu}_2 = \mu_2$, $\tilde{K}_2 = K_2$ in Ω_0.

PROOF. Let $w = u_2(\cdot; d) - u_1(\cdot; d)$. Then, w is a radiation solution of the Helmholtz equation in Ω_c, and by Lemma 2.1, $w = 0$ in Ω_c. Using the definition of weak solutions of the transmission problem, we get

$$\int_\Omega \left[\mu_j \nabla u_j \nabla \varphi - K_j^2 u_j \varphi\right] dx = 0,$$

for all $\varphi \in H_0^1(\Omega)$, $j = 1, 2$, and hence,

$$\int_\Omega \left[-\mu_2 \nabla w \nabla \varphi + K_2^2 w \varphi\right] dx = \int_\Omega \left[(\mu_2 - \mu_1) \nabla u_1 \nabla \varphi + \left(K_1^2 - K_2^2\right) u_1 \varphi\right] dx.$$

Now, we choose another domain $\overline{\Omega}_i \subseteq \Omega$ such that $\overline{\Omega}_0 \subseteq \Omega_i$, and a function $\psi(x) \in C^\infty(\mathbb{R}^3)$ such that $\psi(x) = 1$ when $x \in \Omega_0$ and $\psi(x) = 0$ when $x \in \mathbb{R}^3 \setminus \overline{\Omega}_i$. If $\tilde{v}_2$ is a solution of (2.15), we have

$$\begin{aligned}
&\int_\Omega \left[(\mu_2 - \mu_1) \nabla u_1 \nabla (\psi \tilde{v}_2) + \left(K_1^2 - K_2^2\right) u_1 \psi \tilde{v}_2\right] dx \\
&\quad = \int_\Omega \left[-\mu_2 \nabla w \nabla (\psi \tilde{v}_2) + K_2^2 w \psi \tilde{v}_2\right] dx \\
&\quad = \int_\Omega \left[-\tilde{\mu}_2 \nabla w \nabla \tilde{v}_2 + \tilde{K}_2^2 w \tilde{v}_2\right] dx = 0.
\end{aligned} \tag{2.16}$$

On the other hand, from the variational equation (2.6), we can prove that

$$\int_\Omega \left[(\lambda_j - 1) \chi(D_j) \nabla u_j \nabla \varphi - \left(\lambda_j k_j^2 - k^2\right) \chi(D_j) u_j \varphi\right] dx = 0,$$

for all $\varphi \in H_0^1(\Omega)$, where $\chi(D_j)$ denote the characteristic functions. We note also that

$$\mu_2 - \mu_1 = (\lambda_2 - 1) \chi(D_2) - (\lambda_1 - 1) \chi(D_1), \tag{2.17}$$
$$k_1^2 - k_2^2 = \left(\lambda_1 k_1^2 - k^2\right) \chi(D_1) - \left(\lambda_2 k_2^2 - k^2\right) \chi(D_2). \tag{2.18}$$

From (2.16)–(2.18), we obtain (2.14).

In fact, Lemma 2.3 can be generalized as follows.

LEMMA 2.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\overline{\Omega}_0 \subseteq \Omega$. For all incident directions d, assume that $u_{\infty,1}(\cdot; d) = u_{\infty,2}(\cdot; d)$. Then,*

$$\int_{D_1} \left[(1 - \lambda_1) \nabla \tilde{v}_1 \nabla \tilde{v}_2 + \left(k_1^2 - k^2\right) \tilde{v}_1 \tilde{v}_2\right] dx = \int_{D_2} \left[(1 - \lambda_2) \nabla \tilde{v}_1 \nabla \tilde{v}_2 + \left(k_2^2 - k^2\right) \tilde{v}_1 \tilde{v}_2\right] dx,$$

where $\tilde{v}_j \in H^1(\Omega), j = 1, 2$, are solutions of the variational equations

$$\int_\Omega \left[-\tilde{\mu}_j \nabla \tilde{v}_j \nabla \varphi + \tilde{K}_j^2 \tilde{v}_j \varphi\right] dx = 0, \qquad \varphi \in H_0^1(\Omega),$$

where $\tilde{\mu}_j, \tilde{K}_j \in L^\infty(\Omega)$ satisfy $\tilde{\mu}_j = \mu_j$, $\tilde{K}_j = K_j$ in Ω_0.

The proof combines the method of Lemma 2.3 and the idea of Theorem 3.2 in [6]. We shall omit it.

3. UNIQUENESS

As we are assuming that the boundary ∂D of class C^2, at each point $x_0 \in \partial D$, there exists a neighbourhood V_{x_0} of x_0 such that the intersection $\partial D \cap V_{x_0}$ can be mapped bijectively onto some open domain $U \in \mathbb{R}^2$, and that this mapping is twice continuously differentiable. We describe this mapping in the form

$$\tau(z) = \left(x^1(z_1, z_2), x^2(z_1, z_2), x^3(z_1, z_2)\right),$$

where $\tau : U = \{z \in \mathbb{R}^2, |z| < 1\} \longrightarrow \mathbb{R}^3$ with $\tau(0) = x_0$.

We also consider surfaces ∂D_h parallel to ∂D given by

$$y = x + h\nu(x), \qquad x \in \partial D,$$

where the parameter h denotes the distance of ∂D_h from the generating surface ∂D. We will need the following fact (see [11] for details).

LEMMA 3.1. *Let ∂D be of class of C^2, then the parallel surface ∂D_h is of class C^1 provided the parameter h is sufficiently small.*

Given $D = D_1 = D_2$, and for $j = 1, 2$, we have the transmission problems

$$\begin{aligned} \Delta u_j + k^2 u_j &= 0, && \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \Delta u_j + k_j^2 u_j &= 0, && \text{in } D, \end{aligned}$$

with

$$\begin{aligned} u_j|_+ &= u_j|_-, && \text{on } \partial D, \\ \left.\frac{\partial u_j}{\partial \nu}\right|_+ &= \lambda \left.\frac{\partial u_j}{\partial \nu}\right|_-, && \text{on } \partial D, \end{aligned}$$

where $g|_{+(-)}$ denotes the limit of g on ∂D from the exterior (interior). We have the following uniqueness result.

THEOREM 3.2. *Let k, k_j, and λ_j be the coefficients of the transmission problems for the obstacle $D = D_1 = D_2$. Assume that there are two interior wavenumbers k_1, k_2 for which the far field patterns coincide for all incident directions, i.e., $u_{\infty,1}(\cdot; d) = u_{\infty,2}(\cdot; d)$ for all d. Then, $k_1 = k_2$.*

PROOF. We divide the proof into two steps. First, we construct Green's functions Γ_1, Γ_2 for the transmission problems with interior wavenumbers k_1 and k_2.

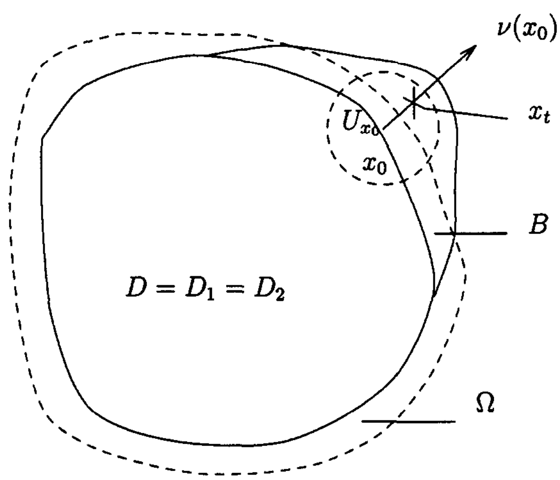

Figure 1.

Choose a point $x_0 \in \partial D$ and a neighbourhood V_{x_0} of x_0. In order to construct uniformly bounded solutions of the transmission problem, we consider a smooth perturbation $\tilde{D}$ of D. Denote by $\tau(z)$ a parametrization of $\partial D \cup V_{x_0}$ with $\tau : \{z \in \mathbb{R}^2 | |z| < 1\} \to \mathbb{R}^3$, $\tau(0) = x_0$. Also, let $\psi \in C^\infty[0,1]$ be a smooth function with $\max_{t \in [0,1]}\{\psi(t)\} = \epsilon > 0$ and

$$\begin{aligned} \psi(t) &> 0, && 0 \le t < \frac{1}{2}, \\ \psi(t) &= 0, && \frac{1}{2} \le t \le 1. \end{aligned}$$

If V_{x_0} and ϵ are sufficiently small, then the domain

$$B = \left\{ \tau(z) + t\psi(|z|)\nu(\tau(z));\ z \in \mathbb{R}^2,\ |z| < \frac{1}{2},\ t \in (0,1) \right\}$$

is well defined. By Lemma 3.1, the boundary of $\overline{D} \cup \overline{B}$ is of class C^1. Write $\tilde{D} = (\overline{D} \cup \overline{B})^0$. By Lemma 2.2, the transmission problem is uniquely solvable for the domain $\tilde{D}$.

Choose a point x in a neighbourhood $U_{x_0} \subseteq \tilde{D}$ of x_0 and consider the following problems:

$$\begin{aligned} \Delta w_j(\cdot;x) + k^2 w_j(\cdot;x) &= 0, \qquad \text{in } \mathbb{R}^3 \setminus \overline{\tilde{D}}, \\ \Delta w_j(\cdot;x) + k_j^2 w_j(\cdot;x) &= 0, \qquad \text{in } \tilde{D}, \end{aligned}$$

with the boundary conditions

$$\begin{aligned} w_j|_+(\cdot;x) &= w_j|_-(\cdot;x) + \Phi_j(\cdot;x), && \text{on } \partial\tilde{D}, \\ \left.\frac{\partial w_j}{\partial \nu}\right|_+(\cdot;x) &= \lambda_j \left.\frac{\partial w_j}{\partial \nu}\right|_-(\cdot;x) + \lambda_j \frac{\partial \Phi_j}{\partial \nu}(\cdot;x), && \text{on } \partial\tilde{D}, \end{aligned}$$

where $D = D_1 = D_2$, w_j are required to satisfy the radiating condition (2.4), and Φ_j are fundamental solutions of the Helmholtz equation with wavenumber k_j.

We choose a domain $\Omega \subseteq \mathbb{R}^3$ with $\overline{D} \subseteq \Omega$ and $x \in (U_{x_0} \bigcap B) \setminus \overline{\Omega}$. Then the functions Γ_1 and Γ_2 can be defined by

$$\Gamma_j(y,x) = \begin{cases} w_j(y,x), & y \in \mathbb{R}^3 \setminus \tilde{D}, \\ w_j(y,x) + \Phi_j(\cdot;x), & y \in \tilde{D} \setminus x. \end{cases}$$

We can prove that $\Gamma_j(y,x)$ is a solution of the variational equation

$$\int_\Omega \left[-\tilde{\mu}_j \nabla_y \Gamma_j(y,x) \nabla\varphi + \tilde{K}_j^2 \Gamma_j(y,x)\varphi \right] dx = 0, \qquad \varphi \in H_0^1(\Omega),$$

where

$$\tilde{\mu}_j(y) = \begin{cases} \mu_j(y), & y \in \mathbb{R}^3 \setminus \overline{B}, \\ \lambda_j, & y \in B, \end{cases}$$

$$\tilde{K}_j^2(y) = \begin{cases} K_j^2(y), & y \in \mathbb{R}^3 \setminus \overline{B}, \\ \lambda_j k_j^2, & y \in B. \end{cases}$$

Now, we apply Lemma 2.4 with $\tilde{v}_1 = \Gamma_1$ and $\tilde{v}_2 = \Gamma_2$.

Note that since $D = D_1 = D_2$,

$$\int_D \left[(\lambda_1 - \lambda_2) \nabla_y \Gamma_1(\cdot;x) \nabla_y \Gamma_2(\cdot;x) + \left(k_2^2 - k_1^2\right) \Gamma_1(\cdot;x)\Gamma_2(\cdot;x) \right] dy = 0,$$

for all $x \in U_{x_0} \setminus \overline{D}$.

From the definition of $\tilde{D}$ and the well-posedness of the transmission problem, we deduce that $w_j \in H^1(\Omega \setminus \overline{\tilde{D}})$ and $w_j \in H^1(\tilde{D})$ are uniformly bounded with respect to $x \in U_{x_0}$. In fact, the functions $w_j(\cdot;x)$ are regular solutions of the corresponding Helmholtz equations in U_{x_0}. By the Sobolev embedding theorem, $w_j(\cdot;x)$ is uniformly bounded in $C^1(U_{x_0})$. If U_{x_0} is sufficiently small, we can prove that

$$\left| \int_{D \cap U_{x_0}} \nabla_y \Gamma_1(\cdot;x_t) \nabla_y \Gamma_2(\cdot;x_t)\, dy \right| \geq C_0 |x_0 - x_t|^{-1},$$

for $x_t = x_0 + t\nu(x_0) \in U_{x_0}$, $t > 0$, and a positive constant C_0. But the other terms of the integral, namely,

$$\int_D \nabla_y \Gamma_1(\cdot; x_t) \nabla_y \Gamma_2(\cdot; x_t)\, dy \quad \text{and} \quad \int_D \Gamma_1(\cdot; x_t)\Gamma_2(\cdot; x_t)\, dy$$

are bounded.

Thus, if $k_1 \neq k_2$, and hence, $\lambda_1 \neq \lambda_2$, we arrive at a contradiction as $t \to 0$. This completes the proof of Theorem 3.2.

REMARK.

(a) The above result is valid for dimension $n \geq 2$.

(b) Based on the same idea, we can show the uniqueness of the scatterer for the transmission problem [1,6,8,10].

REFERENCES

1. D.L. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Heidelberg, (1992).
2. P.D. Lax and R.S. Phillips, *Scattering Theory*, Academic Press, New York, (1967).
3. D.L. Colton and L. Päivärinta, The uniqueness of a solution to an inverse scattering problem for electromagnetic waves, *Arch. Rational Mech. Anal.* **119**, 59–70 (1992).
4. D.L. Colton and B.D. Sleeman, Uniqueness theorems for the inverse problem of acoustic scattering, *IMA J. Appl. Math.* **31**, 253–259 (1983).
5. T. Gerlach and R. Kress, Uniqueness in inverse obstacle scattering with conductive.
6. F. Hettlich, On the uniqueness of the inverse conductive scattering problem for the Helmholtz equation, *Inverse Problems* **10**, 129–144 (1994).
7. V. Isakov, On uniqueness in the inverse transmission scattering problem, *Comm. Part. Diff. Eq.* **15**, 1565–1587 (1990).
8. A. Kirsch and R. Kress, Uniqueness in inverse obstacle scattering, *Inverse Problems* **9**, 285–299 (1993).
9. A. Kirsch, R. Kress, P. Monk and A. Zinn, Two methods for solving the inverse acoustic scattering problem, *Inverse Problems* **4**, 749–770 (1988).
10. A. Kress and G.F. Roach, Transmission problems for the Helmholtz equation, *J. Math. Phys.* **19**, 1433–1437 (1978).
11. D.L. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, Wiley, New York, (1983).
12. A. Kirsch, The domain derivative and two applications in inverse scattering theory, *Inverse Problems* **9**, 81–86 (1993).